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# Robust Sampled Data Eigenstructure Assignment Using the Delta Operator

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**Eigenstructure assignment is considered for a linear time-invariant plant that is represented by the so-called unified delta model, which is valid both for continuous time and sampled data operation of the plant. We show that the eigenvectors of the delta model are identical to the eigenvectors of the continuous time plant and an expression is derived for the eigenstructure assignment feedback gain matrix for the delta model. We extend a sufficient condition for robust stability to the delta model of the plant. A robust sampled data design is computed for the Extended Medium Range Air-to-Air Technology missile by minimizing the integral of the roll rate with constraints on selected eigenvalues, actuator deflection rates, and the sufficient condition for robust stability. The robust design exhibits a significantly improved sideslip gust response as compared to an orthogonal projection eigenstructure assignment design.**

## I. Introduction

RECENTLY, Sobel and Cloutier[1] applied eigenstructure assignment to the design of an autopilot for the Extended Medium Range Air-to-Air Technology (EMRAAT) missile. An important difference between this application and other eigenstructure assignment applications that have appeared in the literature is that the lateral dynamics of the EMRAAT missile does not have a well-defined Dutch roll mode. Therefore, eigenstructure assignment is used not only for mode decoupling but also to create distinctly separate Dutch roll and roll modes. Sobel and Cloutier[1] used the approach suggested by Andry et al.[2] in which the $i$th desired eigenvector $v_i^d$ is chosen for mode decoupling. Then, the $i$th achievable eigenvector $v_i^a$ is chosen as the projection of $v_i^d$ onto the so-called achievability subspace. Sobel and Cloutier[1] showed that their design achieves improved decoupling between an initial sideslip angle and the integrated roll rate (which is approximately equal to the bank angle) when compared to a linear quadratic regulator design proposed by Bossi and Langehough.[3] However, the design of Sobel and Cloutier[1] does not consider that the missile's aerodynamic parameters are uncertain.

A sufficient condition for the robust stability of a linear time-invariant system subject to linear time varying structured state space uncertainty has been proposed by Sobel et al.[4] This result, which is based on the Gronwall lemma, insures robust stability if the nominal eigenvalues lie to the left of a vertical line in the complex plane. This line is determined by the maximum eigenvalue of a matrix that involves the product of the uncertainty structure, the nominal closed-loop modal matrix, and the inverse of the nominal closed-loop modal matrix.

Recently, Yu et al.[5] presented a new sufficient condition for the robust stability of a linear time-invariant system subject to linear time varying structured state space uncertainty. This new robustness condition is a sum of terms each of which involves the $i$th right eigenvector, the $i$th left eigenvector, and the real part of the $i$th eigenvalue. Yu et al. also showed that their robustness condition is less conservative than the earlier result of Sobel et al.[4]

In this paper, we extend eigenstructure assignment to linear time-invariant plants that are represented by Middleton and Goodwin's[6] unified delta model that is valid both for continuous time and sampled data operation of the plant. An important property of the delta model is that the discrete time eigenvalues approach the continuous time eigenvalues as the sampling period approaches zero. We show that the eigenvectors of the delta model are identical to the eigenvectors of the continuous time plant and an expression is derived for the eigenstructure assignment feedback gain matrix for the delta model. We show that, in the limit as the sampling period delta goes to zero, the delta feedback gain approaches the continuous time feedback gain. We propose a sufficient condition for the robust stability of a linear time-invariant unified delta plant subject to linear time-invariant structured state space uncertainty. This yields a new unified robustness condition that is applicable to both continuous time and sampled data operation.

We design a robust controller for the lateral dynamics of the EMRAAT missile and this new design is compared to an orthogonal projection eigenstructure assignment design. The new design method proposed in this paper uses the MATLAB™ Optimization Toolbox[7] and the MATLAB™ Delta Toolbox[8] to minimize the integrated roll rate with constraints on the time constants of the Dutch roll and roll modes, the damping ratios of the Dutch roll and roll modes, the aileron and rudder deflection rates, and the new sufficient condition for robust stability. This design satisfies the new robustness condition while also yielding an improved transient response as compared to the orthogonal projection design.

## II. Problem Formulation

Consider a nominal linear time-invariant multi-input multi-output system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $x \in \Re^n$ is the state vector; $u \in \Re^m$ the input vector; $y \in \Re^r$ the output vector; and $A$, $B$, and $C$ are constant matrices.

The corresponding sampled data system, which is obtained by using Middleton and Goodwin's[6] delta operator, is

$$\delta x = A_\delta x + B_\delta u \tag{3}$$

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$$y = Cx \tag{4}$$

where

$$A_\delta = \Omega A \tag{5}$$

$$B_\delta = \Omega B \tag{6}$$

$$\Omega = \frac{1}{\Delta}\int_0^\Delta e^{A\tau}\,\mathrm{d}\tau \tag{7}$$

and where

$$\delta = (q-1)/\Delta \tag{8}$$

and where the shift operator $q$ is defined by

$$qx_k = x_{k+1} \tag{9}$$

and where $\Delta$ is the sampling period.

The unified state space model proposed by Middleton and Goodwin[6] is valid for both the discrete and continuous time cases simultaneously. This unified model is described by

$$\rho x(t) = A_\rho x(t) + B_\rho u(t) \tag{10}$$

$$y(t) = Cx(t) \tag{11}$$

where

$$A_\delta = \begin{cases} A & \text{in continuous time} \\ A_\delta & \text{in discrete time} \end{cases} \tag{12}$$

$$B_\rho = \begin{cases} B & \text{in continuous time} \\ B_\delta & \text{in discrete time} \end{cases} \tag{13}$$

and where

$$\rho = \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} & \text{in continuous time} \\ \delta & \text{in discrete time} \end{cases} \tag{14}$$

Suppose that the nominal delta system is subject to linear time-invariant uncertainties in the entries of $A_\rho$ and $B_\rho$ described by $\mathrm{d}A_\rho$ and $\mathrm{d}B_\rho$, respectively, where

$$\mathrm{d}A_\rho = \begin{cases} \mathrm{d}A & \text{in continuous time} \\ \mathrm{d}A_\delta & \text{in discrete time} \end{cases} \tag{15}$$

$$\mathrm{d}B_\rho = \begin{cases} \mathrm{d}B & \text{in continuous time} \\ \mathrm{d}B_\delta & \text{in discrete time} \end{cases} \tag{16}$$

and where

$$\mathrm{d}A_\delta = (1/\Delta)\left[e^{(A+\mathrm{d}A)\Delta} - e^{A\Delta}\right] \tag{17}$$

$$\mathrm{d}B_\delta = \frac{1}{\Delta}\left[\int_0^\Delta e^{(A+\mathrm{d}A)\tau}\,\mathrm{d}\tau(B+\mathrm{d}B) - \int_0^\Delta e^{A\tau}\,\mathrm{d}\tau B\right] \tag{18}$$

Then, the delta system with uncertainty is given by

$$\rho x(t) = A_\rho x(t) + B_\rho u(t) + \mathrm{d}A_\rho x(t) + \mathrm{d}B_\rho u(t) \tag{19}$$

$$y(t) = Cx(t) \tag{20}$$

Further, suppose that bounds are available on the maximum absolute values of the elements of $\mathrm{d}A$ and $\mathrm{d}B$. That is,

$$|\mathrm{d}a_{ij}| \le (\mathrm{d}a_{ij})_{\max}; \qquad i = 1,\ldots,n; \qquad j = 1,\ldots,n \tag{21}$$

$$|\mathrm{d}b_{ij}| \le (\mathrm{d}b_{ij})_{\max}; \qquad i = 1,\ldots,n; \qquad j = 1,\ldots,m \tag{22}$$

Then, the corresponding bounds on the $\delta$ system are given by

$$|\mathrm{d}a_\delta(i,j)| \le \left[\mathrm{d}a_\delta(i,j)\right]_{\max}; \quad i = 1,\ldots,n; \quad j = 1,\ldots,n \tag{23}$$

$$|\mathrm{d}b_\delta(i,j)| \le \left[\mathrm{d}b_\delta(i,j)\right]_{\max}; \quad i = 1,\ldots,n; \quad j = 1,\ldots,m \tag{24}$$

Define $\mathrm{d}A_\rho^+$ and $\mathrm{d}B_\rho^+$ as the matrices obtained by replacing the entries of $\mathrm{d}A_\rho$ and $\mathrm{d}B_\rho$ by their absolute values. Also, define $\mathrm{d}A_{\rho\max}$ and $\mathrm{d}B_{\rho\max}$ as the matrices with entries $(\mathrm{d}a_{ij})_{\max}$ and $(\mathrm{d}b_{ij})_{\max}$, respectively, in continuous time or with entries $[\mathrm{d}a_\delta(i,j)]_{\max}$ and $[\mathrm{d}b_\delta(i,j)]_{\max}$, respectively, in discrete time. Then,

$$\left\{\mathrm{d}A_\rho : \mathrm{d}A_\rho^+ \le \mathrm{d}A_{\rho\max}\right\} \tag{25}$$

and

$$\left\{\mathrm{d}B_\rho : \mathrm{d}B_\rho^+ \le \mathrm{d}B_{\rho\max}\right\} \tag{26}$$

where

$$\mathrm{d}A_{\rho\max} = \begin{cases} \mathrm{d}A_{\max} & \text{in continuous time} \\ \mathrm{d}A_{\delta\max} & \text{in discrete time} \end{cases} \tag{27}$$

$$\mathrm{d}B_{\rho\max} = \begin{cases} \mathrm{d}B_{\max} & \text{in continuous time} \\ \mathrm{d}B_{\delta\max} & \text{in discrete time} \end{cases} \tag{28}$$

and where

$$\mathrm{d}A_{\delta\max} = \frac{1}{\Delta}\left[e^{(A^+ + \mathrm{d}A_{\max})\Delta} - e^{A^+\Delta}\right] \tag{29}$$

$$\mathrm{d}B_{\delta\max} = \frac{1}{\Delta}\left[\int_0^\Delta e^{(A^+ + \mathrm{d}A_{\max})\tau}\,\mathrm{d}\tau(B^+ + \mathrm{d}B_{\max}) - \int_0^\Delta e^{A^+\tau}\,\mathrm{d}\tau B^+\right] \tag{30}$$

and where "$\le$" is applied element by element to matrices and $\mathrm{d}A_{\max} \in \mathbb{R}_+^{n\times n}$, $\mathrm{d}B_{\max} \in \mathbb{R}_+^{n\times m}$ where $\mathbb{R}_+$ is the set of nonnegative numbers.

Consider the constant gain output feedback control law described by

$$u(t) = F_\rho y(t) \tag{31}$$

where

$$F_\rho = \begin{cases} F & \text{in continuous time} \\ F_\delta & \text{in discrete time} \end{cases} \tag{32}$$

Then, the nominal closed-loop unified delta system is given by

$$\rho x(t) = A_{\rho c} x(t) \tag{33}$$

where

$$A_{\rho c} = \begin{cases} A + BFC & \text{in continuous time} \\ A_\delta + B_\delta F_\delta C & \text{in discrete time} \end{cases} \tag{34}$$

and the uncertain closed-loop unified delta system is given by

$$\rho x(t) = A_{\rho c} x(t) + \mathrm{d}A_{\rho c} x(t) \tag{35}$$

where

$$\mathrm{d}A_{\rho c} = \begin{cases} \mathrm{d}A + \mathrm{d}B(FC) & \text{in continuous time} \\ \mathrm{d}A_\delta + \mathrm{d}B_\delta(F_\delta C) & \text{in discrete time} \end{cases} \tag{36}$$

Finally, the stability robustness problem can be stated as follows: Given a feedback gain matrix $F_\rho \in \mathbb{R}^{m\times r}$ such that the nominal closed-loop unified delta system exhibits desirable dynamic performance, determine if the uncertain closed-loop unified delta system is asymptotically stable for all time-invariant $dA_\rho$ and $dB_\rho$ described by Eqs. (25) and (26), respectively.

## III. Robustness Results

In this section we present four theorems describing eigenstructure assignment and robust control for the unified delta model. Theorem 1 shows that a matrix $M$ is a modal matrix for the delta plant if and only if it is a modal matrix for the continuous time plant. Theorem 2 describes the settling time and damping ratio regions for the delta plant in the $\gamma$ plane. Theorem 3 describes the eigenstructure assignment output feedback gain matrix for the delta plant and shows that the delta feedback gain matrix approaches the continuous time feedback gain matrix as the sampling period $\Delta$ approaches zero. Theorem 4 presents a sufficient condition for robust stability of the delta plant under time-invariant structured state space uncertainty. This new robustness condition is valid simultaneously for both continuous and discrete time operation of the plant.

*Theorem 1 (Delta Eigenvectors):* Consider the continuous time plant given by $\dot{x}=Ax+Bu$ and the sampled data plant given by $\delta x = A_\delta x + B_\delta u$. The $i$th eigenvalues of $A$ and $A_\delta$ are $\lambda_i$ and $\gamma_i = [\exp(\lambda_i\Delta)-1]/\Delta$, respectively. Let $A$ and $A_\delta$ be nondefective, let $M$ be a modal matrix, and let $\Lambda$ be a diagonal matrix with the $\lambda_i$ on the diagonal. Then,

$$M^{-1}A_\delta M = \gamma_i I = \frac{1}{\Delta}(e^{\Lambda\Delta}-I)$$

$$\text{if and only if} \quad M^{-1}AM = \lambda_i I = \Lambda \tag{37}$$

*Proof:* See Appendix A.

*Theorem 2 (Delta Settling Time and Damping Regions):* Consider the plant described in Theorem 1 with eigenvalues $\lambda_i$ in continuous time and eigenvalues $\gamma_i$ for sampled data operation. The $s$-plane settling time region described by

$$\mathrm{Re}\lambda_i < \sigma \tag{38}$$

maps into the $\gamma$-plane region described by

$$|1+\Delta\gamma_i| < e^{\sigma\Delta} \tag{39}$$

and the $s$-plane damping region described by

$$\cos\left\{\tan^{-1}\left[\mathrm{Im}\lambda_i/(-\mathrm{Re}\lambda_i)\right]\right\} > \zeta \tag{40}$$

maps into the $\gamma$-plane region described by

$$|1+\Delta\gamma_i| < \exp\left[-\zeta\phi/(1-\zeta^2)^{1/2}\right] \tag{41}$$

where $\phi = \arg(1+\Delta\gamma)$.

*Proof:* See Appendix B.

*Theorem 3 (Delta Eigenstructure Assignment Feedback Gain Matrix):* Suppose $u=F_\delta y$ with $F_\delta \in \mathbb{R}^{m\times r}$ such that the nominal closed-loop system $A_\delta + B_\delta F_\delta C$ is asymptotically stable with $A_\delta + B_\delta F_\delta C$ nondefective and $r \geq m$. Let $\Lambda_{\delta r}$ be the $r\times r$ diagonal matrix whose entries are the assignable closed-loop eigenvalues and let $M_r$ be the $n\times r$ matrix whose columns are the corresponding achievable eigenvectors. Then, the solution to

$$(A_\delta + B_\delta F_\delta C)M_r = M_r\Lambda_{\delta r} \tag{42}$$

is given by

$$F_\delta = V_\delta \Sigma_\delta^{-1} U_{\delta 0}^T (M_r\Lambda_{\delta r} - A_\delta M_r) V_r \Sigma_r^{-1} U_{r0}^T \tag{43}$$

where

$$B_\delta = [U_{\delta 0} \quad U_{\delta 1}]\begin{bmatrix}\Sigma_\delta V_\delta^T \\ 0\end{bmatrix} \tag{44}$$

and

$$CM_r = [U_{r0} \quad U_{r1}]\begin{bmatrix}\Sigma_r V_r^T \\ 0\end{bmatrix} \tag{45}$$

are the singular value decompositions of $B_\delta$ and $CM_r$, respectively. Furthermore, when $\Delta\to 0$, $F_\delta \to F$ where

$$F = V_B \Sigma_B^{-1} U_{B0}^T (M_r\Lambda_r - AM_r) V_r \Sigma_r^{-1} U_{r0}^T \tag{46}$$

is the feedback gain for the continuous time plant that satisfies

$$(A+BFC)M_r = M_r\Lambda_r \tag{47}$$

and where

$$B = [U_{B0} \quad U_{B1}]\begin{bmatrix}\Sigma_B V_B^T \\ 0\end{bmatrix} \tag{48}$$

is a singular value decomposition of $B$, and $\Lambda_r$ is a diagonal $r\times r$ matrix containing the assignable eigenvalues $\lambda_i$ $(i=1,2,\ldots,r)$.

*Proof:* See Appendix C.

*Theorem 4 (Unified Robustness Sufficient Condition):* Suppose that $F_\rho$ is such that the nominal closed-loop system described by Eq. (33) is asymptotically stable with $A_{\rho c}$ a nondefective matrix. Then the uncertain closed-loop system given by Eq. (35) is asymptotically stable for $dA$ and $dB$ described by Eqs. (25) and (26), respectively, if

$$\lambda_{\max}\left\{\sum_{i=1}^{n}\frac{(v_i w_i^*)^+}{f(\xi_1)}\, dA_{\rho c\max}\right\} < 1 \tag{49}$$

where

$$f(\xi_i) = \begin{cases} -\mathrm{Re}(\lambda_i) & \text{continuous time} \\ \dfrac{1}{\Delta}\left[1-(1+\Delta\gamma_i)^+\right] & \text{discrete time}\end{cases} \tag{50}$$

$$dA_{\rho c\max} = \begin{cases} dA_{\max} + dB_{\max}(FC)^+ & \text{continuous time} \\ dA_{\delta\max} + dB_{\delta\max}(F_\delta C)^+ & \text{discrete time}\end{cases} \tag{51}$$

$$dA_{\delta\max} = \frac{1}{\Delta}\left[e^{(A^+ + dA_{\max})\Delta} - e^{A^+\Delta}\right]$$

$$dB_{\delta\max} = \frac{1}{\Delta}\left[\int_0^\Delta e^{(A^+ + dA_{\max})\tau}\, d\tau (B^+ + dB_{\max}) - \int_0^\Delta e^{A^+\tau}\, d\tau B^+\right] \tag{52}$$

and where $\xi_i$ is the $i$th eigenvalue of $(A_\rho + B_\rho F_\rho C)$ with $v_i$ and $w_i^*$ the corresponding right and left eigenvectors, respectively; and where $(\cdot)^*$ denotes the complex conjugate transpose.

*Proof:* See Appendix D.

## IV. Robust Control Design for the EMRAAT Missile

Consider the EMRAAT bank-to-turn missile that is described by Bossi and Langehough.[3] A seventh-order model of the yaw/roll dynamics at a 10-deg angle of attack is considered, which includes the rigid-body dynamics, two first-order actuator models, and a yaw rate washout filter. The state vector, control vector, and measurement vector are given, respectively, by

$$x = [\beta, r, p, p_I, \delta_r, \delta_a, x_7]^T, \quad u = [\delta_{rc}, \delta_{ac}]^T, \quad y = [\beta, r_{\mathrm{wo}}, p, p_I]^T$$

**Table 1 Comparison of EMRAAT designs, $\Delta = 0.0025$ s**

| Closed-loop eigenvalues[a] | | Feedback gain matrix |
|---|---|---|
| Orthogonal projection design | $\gamma_{dr} = -23.65 \pm j\,16.97$<br>$\gamma_{roll} = -9.98 \pm j\,10.16$<br>$\gamma_{act} = -130.3$<br>$\gamma_{act} = -103.8$<br>$\gamma_{filter} = -6.97$ | $\begin{array}{ccccc} \beta & r_{wo} & p & p_I & \\ -5.43 & 0.231 & 0.0043 & 0.959 & \delta_{rc} \\ 4.42 & -0.285 & 0.0050 & -1.09 & \delta_{ac} \end{array}$ |
| Robust design | $\gamma_{dr} = -18.26 \pm j\,14.15$<br>$\gamma_{roll} = -61.39 \pm j\,99.70$<br>$\gamma_{act} = -112.8$<br>$\gamma_{act} = -48.61$<br>$\gamma_{filter} = -7.42$ | $\begin{array}{ccccc} \beta & r_{wo} & p & p_I & \\ -4.16 & 0.196 & 0.0154 & 1.27 & \delta_{rc} \\ 2.36 & -0.215 & 0.0757 & 2.16 & \delta_{ac} \end{array}$ |

[a]Eigenvalues are computed by using feedback gains which are rounded to three significant digits.

**Table 2 Eigenvectors for the EMRAAT designs[a]**

Desired closed-loop eigenvectors:

$$\underbrace{\begin{bmatrix} 1 \\ x \\ 0 \\ 0 \\ x \\ x \\ x \end{bmatrix} \pm j \begin{bmatrix} x \\ 1 \\ 0 \\ 0 \\ x \\ x \\ x \end{bmatrix}}_{\text{Dutch roll mode}} \qquad \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ x \\ x \\ x \\ 0 \end{bmatrix} \pm j \begin{bmatrix} 0 \\ 0 \\ x \\ 1 \\ x \\ x \\ 0 \end{bmatrix}}_{\text{Roll mode}} \begin{array}{c} \beta \\ r \\ p \\ p_I \\ \delta_r \\ \delta_a \\ x_7 \end{array}$$

Orthogonal projection design:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.0258 | 0.0203 | −0.0083 | −0.0092 | −0.0012 | 0.0074 | 0.0327 | $\beta$ |
| 1.0000 | 0.0000 | 0.0000 | −0.0003 | −0.0197 | 1.0000 | 0.3931 | $r$ |
| −0.0006 | 0.0001 | 1.0000 | 0.0000 | 1.0000 | 0.1780 | 1.0000 | $p$ |
| 0.0000 | 0.0000 | −0.0480 | −0.0501 | −0.0063 | −0.0015 | −0.1423 | $p_I$ |
| 0.1309 | −0.1140 | 0.0037 | 0.0016 | 0.0024 | 0.7135 | 0.0053 | $\delta_r$ |
| −0.2382 | 0.0960 | 0.0231 | 0.0134 | 0.1258 | −0.9342 | −0.0880 | $\delta_a$ |
| −0.1385 | −0.1316 | −0.0001 | 0.0000 | 0.0006 | −0.0434 | −0.9699 | $x_7$ |
| Re $v_{dr}$ | Im $v_{dr}$ | Re $v_{roll}$ | Im $v_{roll}$ | Actuator | Actuator | Filter | |

Robust design:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.0312 | 0.0280 | −0.0006 | −0.0013 | −0.0034 | −0.0070 | −0.0693 | $\beta$ |
| 1.0000 | 0.0000 | −0.0003 | −0.0002 | −0.0005 | −0.9184 | −0.4987 | $r$ |
| 0.0038 | −0.1691 | 1.0000 | 0.0000 | 1.0000 | 0.4932 | −0.0468 | $p$ |
| −0.0046 | 0.0055 | −0.0032 | −0.0073 | −0.0193 | −0.0037 | 0.0062 | $p_I$ |
| 0.0958 | −0.0977 | 0.0063 | −0.0145 | 0.0075 | −0.7119 | 0.0016 | $\delta_r$ |
| −0.2083 | 0.0524 | 0.0319 | −0.0711 | 0.0393 | 1.0000 | 0.1779 | $\delta_a$ |
| −0.1678 | −0.1854 | 0.0000 | 0.0000 | 0.0001 | 0.0360 | 1.0000 | $x_7$ |
| Re $v_{dr}$ | Im $v_{dr}$ | Re $v_{roll}$ | Im $v_{roll}$ | Actuator | Actuator | Filter | |

[a]Eigenvalues are computed by using feedback gains which are rounded to three significant digits.

Here $\beta$ is the sideslip angle, deg; $r$ the yaw rate, deg/s; $p$ the roll rate, deg/s; $p_I$ the integrated roll rate, deg; $\delta_r$ the rudder deflection, deg; $\delta_a$ the aileron deflection, deg; $x_7$ the yaw rate washout filter state; and $r_{wo}$ the washed out yaw rate, deg/s. The state space matrices $A$, $B$, and $C$ are shown in Appendix E.

First, we design an eigenstructure assignment control law by using an orthogonal projection. The delta state space matrices $A_\delta$ and $B_\delta$ are computed by using the MATLAB™ Delta Toolbox.[8] The sampling period $\Delta$ is chosen to be 0.0025 s for illustrative purposes. The desired Dutch roll and roll mode eigenvalues are achieved exactly because four measurements are available for feedback. The desired Dutch roll eigenvectors are chosen to yield a complex mode that is composed of sideslip angle and yaw rate with no coupling to roll rate and integrated roll rate. The desired roll mode eigenvectors are chosen to yield a complex mode that is composed of roll rate and integrated roll rate (which is approximately equal to the bank angle) with no coupling to sideslip angle and yaw rate. Then, the achievable eigenvectors are computed by using the orthogonal projection of the $i$th desired eigenvector $v_i^d$ onto the subspace that is spanned by the columns of $(\gamma_i I - A_\delta)^{-1} B_\delta$. The closed-loop delta eigenvalues $\gamma_i$, $i = 1, \ldots, n$, and the feedback gain matrix $F_\delta$ are shown in Table 1. The desired and achievable closed-loop eigenvectors are shown in Table 2.

Next, we propose a new robust design that minimizes the integrated roll rate due to a 1-deg initial sideslip angle subject to constraints on the time constants of the Dutch roll and roll

**Table 3 Constraints for the EMRAAT designs**

| Continuous time | Discrete time |
|---|---|
| $\mathrm{Re}\lambda_{dr} \in [-50, -6]$ | $\lvert 1+\Delta\gamma_{dr}\rvert \in [e^{-50\Delta}, e^{-6\Delta}]$ |
| $\mathrm{Re}\lambda_{roll} \in [-50, -6]$ | $\lvert 1+\Delta\gamma_{roll}\rvert \in [e^{-50\Delta}, e^{-6\Delta}]$ |
| $\lambda_{rudder} < -50$ | $1+\Delta\gamma_{rudder} < e^{-50\Delta}$ |
| $\lambda_{aileron} < -50$ | $1+\Delta\gamma_{aileron} < e^{-50\Delta}$ |
| $\zeta_{dr} \in [0.4, 0.8]$ | $\lvert 1+\Delta\gamma_{dr}\rvert \in \left[\exp\left(\frac{-0.8\phi_{dr}}{[1-(0.8)^2]^{1/2}}\right), \exp\left(\frac{-0.4\phi_{dr}}{[1-(0.4)^2]^{1/2}}\right)\right]$ where $\phi_{dr}=\arg(1+\Delta\gamma_{dr})$ |
| $\zeta_{roll} \in [0.4, 0.8]$ | $\lvert 1+\Delta\gamma_{roll}\rvert \in \left[\exp\left(\frac{-0.8\phi_{roll}}{[1-(0.8)^2]^{1/2}}\right), \exp\left(\frac{-0.4\phi_{roll}}{[1-(0.4)^2]^{1/2}}\right)\right]$ where $\phi_{roll}=\arg(1+\Delta\gamma_{roll})$ |
| $\lvert\dot{\delta}_a\rvert < 275$ deg/s | $\frac{\lvert\delta_a[(k+1)\Delta]-\delta_a(k\Delta)\rvert}{\Delta} < 275$ deg/s |
| $\lvert\dot{\delta}_r\rvert < 275$ deg/s | $\frac{\lvert\delta_r[(k+1)\Delta]-\delta_r(k\Delta)\rvert}{\Delta} < 275$ deg/s |
| $\lambda_{\max}\left\{\sum_{i=1}^{7}\frac{(v_iw_i^*)^+}{\alpha_i}\left[A_{\max}+B_{\max}(FC)^+\right]\right\} < 0.999$ | $\lambda_{\max}\left\{\sum_{i=1}^{7}\frac{(v_iw_i^*)^+}{f(\gamma_i)}\left[A_{\delta\max}+B_{\delta\max}(F_\delta C)^+\right]\right\} < 0.999$ |

modes, the damping ratios of the Dutch roll and roll modes, the aileron and rudder deflection rates, and the new sufficient condition for robust stability. Mathematically, the objective function to be minimized is given by

$$J = \sum_{k=1}^{120}\left[p_I(k\Delta)\right]^2 \tag{53}$$

The upper limit on the index $k$ is chosen to include the time interval $k\Delta \in [0,0.3]$ during which most of the transient response occurs. Of course, computation of Eq. (53) requires that the linear closed-loop response be calculated during each iteration of the optimization algorithm, using the gain that results from the existing values of the optimizing parameters. The constraints for continuous time and the corresponding constraints for discrete time are shown in Table 3 where $\zeta$ is the damping ratio.

For illustrative purposes we have chosen $dA_{\max}=0.04\cdot A^+$ and $dB_{\max}=0$ with the exception that the elements of $dA_{\max}$ that correspond to actuator or washout filter time constants have been set to zero. Also, the coefficients in the equation $\dot{p}_I(t)=p(t)$ have no uncertainty. The matrices $dA_{\delta\max}$ and $dB_{\delta\max}$ are computed from Eqs. (29) and (30), respectively.

The actuator deflection rates are computed from the slopes of the time response of the deflections during the time interval $m\Delta \in [0,0.03]$. This interval is chosen because the slopes of the deflections are largest during this time interval. The maximum slope is chosen from all of the slopes that are calculated between successive pairs of points. Mathematically,

$$\text{max slope } \delta_a = \max_m \frac{\lvert\delta_a[(m+1)\Delta]-\delta_a(m\Delta)\rvert}{\Delta} \tag{54}$$

$$\text{max slope } \delta_r = \max_m \frac{\lvert\delta_r[(m+1)\Delta]-\delta_r(m\Delta)\rvert}{\Delta} \tag{55}$$

where $m=0,1,\ldots,12$. These slopes are recomputed during each iteration of the algorithm. The maximum deflection rates chosen for the constraints are well within the expected 400-deg/s limit for the advanced state of the art electromechanical actuator described by Langehough and Simons.[9]

The parameter vector contains the quantities that may be varied by the optimization. This 12-dimensional vector includes $\mathrm{Re}\gamma_{dr}$, $\mathrm{Im}\gamma_{dr}$, $\mathrm{Re}\gamma_{roll}$, $\mathrm{Im}\gamma_{roll}$, $\mathrm{Re}z_1(1)$, $\mathrm{Re}z_1(2)$, $\mathrm{Im}z_1(1)$, $\mathrm{Im}z_1(2)$, $\mathrm{Re}z_3(1)$, $\mathrm{Re}z_3(2)$, $\mathrm{Im}z_3(1)$, and $\mathrm{Im}z_3(2)$. Here, the two-dimensional complex vectors $z_i$ contain the free eigenvector parameters. That is, the $i$th eigenvector $v_i$ may be written as

$$v_i = L_i z_i \tag{56}$$

where the columns of $L_i=(\gamma_i I - A_\delta)^{-1}B_\delta$ are a basis for the subspace in which the $i$th eigenvector must reside. Thus, the free parameters are the vectors $z_i$ rather than the eigenvectors $v_i$.

The optimization is performed by using subroutine *constr* from the MATLAB™ Optimization Toolbox[7] and subroutine *delsim* from the MATLAB™ Delta Toolbox[8] on a 486™ 25 MHz personal computer. The optimization is initialized with the orthogonal projection design, which yields an initial value of 9.1360 for the objective function of Eq. (53) and a value of 2.2014 for the left-hand side (LHS) of the robustness condition of Eq. (49). The optimization is complete after 3640 iterations and yields an optimal objective function of 0.0895 and a value of 0.999 for the LHS of the robustness condition. We observe from Table 1 that the Dutch roll mode is dominant in the robust design whereas the roll mode was chosen to be dominant in the orthogonal projection design. Furthermore, the optimization moves the roll mode eigenvalues to the boundary of the feasible set that corresponds to the smallest time constant and the smallest damping ratio allowed by the constraints. We observe from Table 2 that the yaw rate washout filter eigenvector for the robust design is characterized by a significant reduction in the roll rate and integrated roll rate entries as compared with the orthogonal projection design. However, the optimized Dutch roll eigenvectors exhibit an increase in the roll rate and integrated roll rate entries which were desired to be zero in the orthogonal projection design. We conjecture that the optimization alters the filter eigenvector to improve mode decoupling whereas the Dutch roll mode eigenvector is altered to satisfy the robustness constraint.

The aircraft responses to a "1-cosine" sideslip gust, as described in Ref. 10, are considered. The state equations for the lateral dynamics are shown by McRuer et al.[11] to be given by

$$\dot{\beta} = Y_v\beta + (g/U_0)\phi - r + (Y_{\delta_a}/U_0)\delta_a + (Y_{\delta_r}/U_0)\delta_r - Y_v\beta_g \tag{57}$$

$$\dot{p} = L'_\beta \beta + L'_p p + L'_r r + L'_{\delta_a} \delta_a + L'_{\delta_r} \delta_r - L'_\beta \beta_g - (L'_r)_g \dot{\beta}_g \tag{58}$$

$$\dot{r} = N'_\beta \beta + N'_p p + N'_r r + N'_{\delta_a} \delta_a + N'_{\delta_r} \delta_r - N'_\beta \beta_g - (N'_r)_g \dot{\beta}_g \tag{59}$$

where $Y_v$, $Y_v/U_0$, $Y_{\delta_a}/U_0$, $Y_{\delta_r}/U_0$, $L'_\beta$, $L'_p$, $L'_r$, $L'_{\delta_a}$, $L'_{\delta_r}$, $N'_\beta$, $N'_p$, $N'_r$, $N'_{\delta_a}$, and $N'_{\delta_r}$ can be obtained from the state space matrix $A$ and where $(N'_r)_g$ and $(L'_r)_g$ are defined in Ref. 11 to be

$$(N'_r)_g = N_r\left[1 - \frac{I_{xz}^2}{I_x I_z}\right]^{-1} \tag{60}$$

$$(L'_r)_g = \left(\frac{I_{xz}}{I_x}\right) N_r \left[1 - \frac{I_{xz}^2}{I_x I_z}\right]^{-1} \tag{61}$$

$$N_r = N'_r - \frac{I_{xz}}{I_z} L'_r \tag{62}$$

For the flight condition of the EMRAAT missile considered in this paper, the parameters $N'_r$ and $L'_r$ are $-0.5748$ and 0.3208, respectively. The inertias corresponding to a full fuel condition are $I_x = 11{,}451$; $I_z = 456{,}282$; and $I_{xz} = -1189$. Using these values, the gust derivative coefficients are computed to be $(N'_r)_g = -0.5742$ and $(L'_r)_g = 0.05962$. The gust is defined as shown in Ref. 10 and is described by

$$\beta_g = \begin{cases} 0 & t < 0 \\ 0.5(1 - \cos 24\pi t) & 0 \le t \le 1/24 \\ 1 & t > 1/24 \end{cases} \tag{63}$$

where the natural frequency of the open-loop complex eigenvalue pair is 24.04 rad/s.

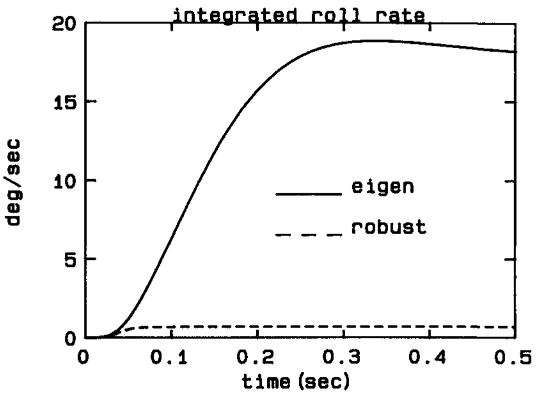


**Fig. 1 Integrated roll rate.**

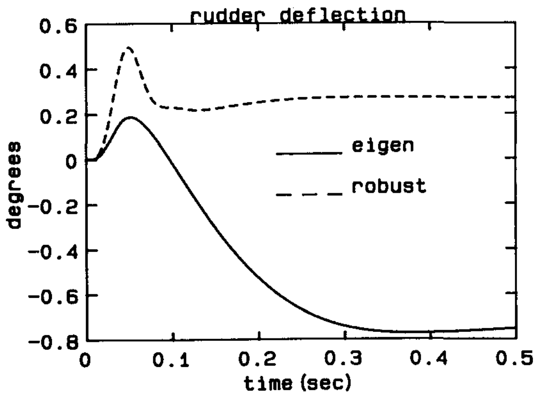


**Fig. 2 Rudder deflection.**

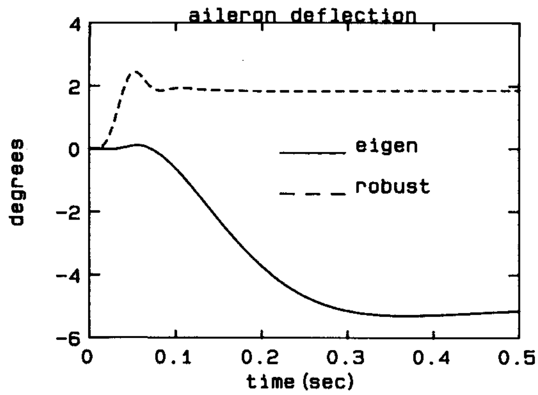


**Fig. 3 Aileron deflection.**

The time histories of integrated roll rate, rudder deflection, and aileron deflection to the 1-cosine sideslip gust are shown in Figs. 1, 2, and 3, respectively. We observe a significant improvement in the integrated roll rate response (which is desired to be zero) when compared to the initial orthogonal projection eigenstructure assignment design. The initial design has a maximum $p_I(t)$ of 18.87 deg, but the new design of this paper has a maximum $p_I(t)$ of 0.6990 deg that is an improvement of approximately 96%. We note that this improved response is obtained with both smaller aileron and rudder deflections.

## V. Conclusions

We have extended eigenstructure assignment to linear time-invariant plants that are represented by the so-called unified delta model which is valid both for continuous time and sampled data operation of the plant. We have shown that the eigenvectors of the delta model are identical to the eigenvectors of the continuous time plant and an expression was derived for the eigenstructure assignment feedback gain matrix for the delta model. We have proposed a sufficient condition for the robust stability of a linear time-invariant unified delta plant subject to linear time-invariant structured state space uncertainty. A robust sampled data design is computed for the Extended Medium Range Air-to-Air Technology missile by minimizing the integral of the roll rate with constraints on selected eigenvalues, actuator deflection rates, and the sufficient condition for robust stability. The robust design is compared with an orthogonal projection eigenstructure assignment design. In particular, the maximum integrated roll rate to a 1-cosine sideslip gust is reduced from 18.87 deg for the orthogonal projection design to 0.6990 deg for the robust design.

## Appendix A: Proof of Theorem 1

Sufficiency: Use the definition of $A_\delta$ to obtain

$$M^{-1} A_\delta M = \frac{1}{\Delta} M^{-1}(e^{A\Delta} - I)M \tag{A1}$$

Substitute the infinite series for $\exp(A\Delta)$ into Eq. (A1) to obtain

$$M^{-1} A_\delta M = \frac{1}{\Delta} M^{-1}\left[I + A\Delta + \frac{A^2\Delta^2}{2!} + \cdots - I\right] M \tag{A2}$$

$$M^{-1} A_\delta M = \frac{1}{\Delta}\left[I + M^{-1}AM\Delta + \frac{M^{-1}A^2M\Delta^2}{2!} + \cdots - I\right] \tag{A3}$$

$$M^{-1} A_\delta M = \frac{1}{\Delta}\left[I + M^{-1}AM\Delta + (M^{-1}AM)^2 \frac{\Delta^2}{2!} + (M^{-1}AM)^3 \frac{\Delta^3}{3!} + \cdots - I\right] \tag{A4}$$

Substitute $M^{-1}AM = \Lambda$ into Eq. (A4) to obtain

$$M^{-1}A_\delta M = \frac{1}{\Delta}\left[I + \Lambda\Delta + \Lambda^2\frac{\Delta^2}{2!} + \Lambda^3\frac{\Delta^3}{3!} + \cdots - I\right] \tag{A5}$$

$$= \frac{1}{\Delta}(e^{\Lambda\Delta} - I) \tag{A6}$$

□

Necessity:

$$M^{-1}A_\delta M = \frac{1}{\Delta}M^{-1}(e^{A\Delta} - I)M = \frac{1}{\Delta}(e^{\Lambda\Delta} - I) \tag{A7}$$

$$\Rightarrow M^{-1}e^{A\Delta}M = e^{\Lambda\Delta} \tag{A8}$$

$$\Rightarrow e^{M^{-1}AM\Delta} = e^{\Lambda\Delta} \tag{A9}$$

$$\Rightarrow M^{-1}AM\Delta = \Lambda\Delta \tag{A10}$$

$$\Rightarrow M^{-1}AM = \Lambda \tag{A11}$$

□

## Appendix B: Proof of Theorem 2

Settling time region: Use $1+\Delta\gamma_i = \exp(\lambda_i\Delta)$ to obtain $|1+\Delta\gamma_i| = \exp(\Delta\cdot\mathrm{Re}\lambda_i)$. Then, $\mathrm{Re}\lambda_i < \sigma \Rightarrow |1+\Delta\gamma_i| < \exp(\sigma\Delta)$.

Damping ratio region:

$$\lambda_i = -\zeta_i\omega_{ni} + j\omega_{di} \tag{B1}$$

$$1 + \Delta\gamma_i = \exp(\lambda_i\Delta) = \exp(-\zeta_i\omega_{ni}\Delta + j\omega_{di}\Delta) \tag{B2}$$

Use $\omega_s\Delta = 2\pi$ and $\omega_{di} = \omega_{ni}(1-\zeta_1^2)^{1/2}$ to obtain

$$1 + \Delta\gamma_i = \exp\left[\frac{-\zeta_i 2\pi}{i}(1-\zeta_i^2)^{1/2}\frac{\omega_{di}}{\omega_s} + j2\pi\frac{\omega_{di}}{\omega_s}\right] \tag{B3}$$

which has a magnitude given by

$$|1+\Delta\gamma_i| = \exp\left[\frac{-\zeta_i 2\pi}{(1-\zeta_i^2)^{1/2}}\frac{\omega_{di}}{\omega_s}\right] = \exp\left[\frac{-\zeta_i\phi_i}{(1-\zeta_i^2)^{1/2}}\right] \tag{B4}$$

and a phase angle given by

$$\arg(1+\Delta\gamma_i) = (2\pi\omega_{di}/\omega_s) \tag{B5}$$

Thus,

$$\zeta > \zeta_i \Rightarrow |1+\Delta\gamma_i| < \exp\left[\frac{-\zeta\phi}{(1-\zeta^2)^{1/2}}\right] \tag{B6}$$

where $\phi = \arg(1+\Delta\gamma)$. □

## Appendix C: Proof of Theorem 3

The $r$ assignable eigenvalues and their corresponding eigenvectors for the sampled data system satisfy the equations described by

$$(A_\delta + B_\delta F_\delta C)v_i = \gamma_i v_i; \qquad i = 1, 2, \ldots, r \tag{C1}$$

Combining the $r$ equations from Eq. (C1) yields

$$(A_\delta + B_\delta F_\delta C)M_r = M_r\Lambda_{\delta r} \tag{C2}$$

Rearrange to obtain

$$B_\delta F_\delta C M_r = M_r\Lambda_{\delta r} - A_\delta M_r \tag{C3}$$

Substitute the singular value decompositions of $B_\delta$ and $CM_r$ to obtain

$$[U_{\delta 0} \quad U_{\delta 1}]\begin{bmatrix}\Sigma_\delta V_\delta^T\\ 0\end{bmatrix}F_\delta[U_{r0} \quad U_{r1}]\begin{bmatrix}\Sigma_r V_r^T\\ 0\end{bmatrix} = M_r\Lambda_{\delta r} - A_\delta M_r \tag{C4}$$

$$U_{\delta 0}\Sigma_\delta V_\delta^T F_\delta U_{r0}\Sigma_r V_r^T = M_r\Lambda_{\delta r} - A_\delta M_r \tag{C5}$$

Taking the required inverses and recalling that $U_{\delta 0}$, $U_{r0}$, $V_\delta$, and $V_r$ are unitary, we obtain

$$F_\delta = V_\delta\Sigma_\delta^{-1}U_{\delta 0}^T(M_r\Lambda_{\delta r} - A_\delta M_r)V_r\Sigma_r^{-1}U_{r0}^T \tag{C6}$$

Next, we consider the limiting behavior of $F_\delta$ as $\Delta\to 0$. Middleton and Goodwin[6] show that as $\Delta\to 0$, $A_\delta\to A$, $B_\delta\to B$, and $\Lambda_{\delta r}\to\Lambda_r$. Thus, $V_\delta\Sigma_\delta^{-1}U_{\delta 0}^T\to V_B\Sigma_B^{-1}U_{B0}^T$, and $F_\delta\to F$. □

## Appendix D: Proof of Theorem 4

The proof starts from showing the following lemma.

*Lemma D1 (Delta Uncertainty Bounds):*

$$(\mathrm{d}A_\delta)^+ \le \mathrm{d}A_{\delta\max} = \frac{1}{\Delta}\left[e^{(A^+ + \mathrm{d}A_{\max})\Delta} - e^{A^+\Delta}\right] \tag{D1}$$

$$(\mathrm{d}B_\delta)^+ \le \mathrm{d}B_{\delta\max} = \frac{1}{\Delta}\left[\int_0^\Delta e^{(A^+ + \mathrm{d}A_{\max})\tau}\,\mathrm{d}\tau(B^+ + \mathrm{d}B_{\max}) - \int_0^\Delta e^{A^+\tau}\,\mathrm{d}\tau B^+\right] \tag{D2}$$

*Proof of Lemma D1:*

$$(\mathrm{d}A_\delta)^+ = \frac{1}{\Delta}\left[e^{(A+\mathrm{d}A)\Delta} - e^{A\Delta}\right]^+ = \frac{1}{\Delta}\left\{\left[I + (A+\mathrm{d}A)\Delta + (A+\mathrm{d}A)^2\frac{\Delta^2}{2!} + (A+\mathrm{d}A)^3\frac{\Delta^3}{3!} + \cdots\right] - \left[I + A\Delta + A^2\frac{\Delta^2}{2!} + A^3\frac{\Delta^3}{3!} + \cdots\right]\right\}^+$$

$$= \frac{1}{\Delta}\left\{\mathrm{d}A\cdot\Delta + \left[A\cdot\mathrm{d}A + \mathrm{d}A\cdot A + (\mathrm{d}A)^2\right]\frac{\Delta^2}{2!} + \left[A\cdot\mathrm{d}A\cdot A + \mathrm{d}A\cdot A\cdot\mathrm{d}A + \mathrm{d}A\cdot A^2 + A^2\cdot\mathrm{d}A + (\mathrm{d}A)^2\cdot A + A\cdot(\mathrm{d}A)^2 + (\mathrm{d}A)^3\right]\frac{\Delta^3}{3!}\right.$$

$$\left. + \cdots\right\}^+ \le \frac{1}{\Delta}\left\{\mathrm{d}A_{\max}\cdot\Delta + \left[A^+\cdot\mathrm{d}A_{\max} + \mathrm{d}A_{\max}\cdot A^+ + (\mathrm{d}A_{\max})^2\right]\frac{\Delta^2}{2!} + \left[A^+\cdot\mathrm{d}A_{\max}\cdot A^+ + \mathrm{d}A_{\max}\cdot A^+\cdot\mathrm{d}A_{\max} + \mathrm{d}A_{\max}\cdot(A^+)^2\right.\right.$$

$$\left.\left. + (A^+)^2\cdot\mathrm{d}A_{\max} + (\mathrm{d}A_{\max})^2\cdot A + A\cdot(\mathrm{d}A_{\max})^2 + (\mathrm{d}A_{\max})^3\right]\frac{\Delta^3}{3!} + \cdots\right\} = \frac{1}{\Delta}\left\{\left[I + (A^+ + \mathrm{d}A_{\max})\Delta + (A^+ + \mathrm{d}A_{\max})^2\frac{\Delta^2}{2!}\right.\right.$$

$$\left.\left. + (A^+ + \mathrm{d}A_{\max})^3\frac{\Delta^3}{3!} + \cdots\right] - \left[I + A^+\Delta + (A^+)^2\frac{\Delta^2}{2!} + (A^+)^3\frac{\Delta^3}{3!} + \cdots\right]\right\} = \frac{1}{\Delta}\left[e^{(A^+ + \mathrm{d}A_{\max})\Delta} - e^{A^+\Delta}\right]$$

$$(dB_\delta)^+ = \left\{\frac{1}{\Delta}\left[\int_0^\Delta e^{(A+dA)\tau}\,d\tau(B+dB) - \int_0^\Delta e^{A\tau}\,d\tau B\right]\right\}^+ = \frac{1}{\Delta}\left\{\left[I\Delta+(A+dA)\frac{\Delta^2}{2!}+(A+dA)^2\frac{\Delta^3}{3!}+(A+dA)^3\frac{\Delta^4}{4!}+\cdots\right](B+dB)\right.$$
$$\left.-\left[I\Delta+A\frac{\Delta^2}{2!}+A^2\frac{\Delta^3}{3!}+A^3\frac{\Delta^4}{4!}+\cdots\right]B\right\}^+ = \frac{1}{\Delta}\left\{\left[I\Delta+A\frac{\Delta^2}{2!}+A^2\frac{\Delta^3}{3!}+A^3\frac{\Delta^4}{4!}+\cdots\right]dB\right.$$
$$+\left[dA\cdot\frac{\Delta^2}{2!}+\left(A\cdot dA+dA\cdot A+(dA)^2\right)\frac{\Delta^3}{3!}+\left(A\cdot dA\cdot A+dA\cdot A\cdot dA+dA\cdot A^2+A^2\cdot dA+(dA)^2\cdot A+A\cdot(dA)^2+(dA)^3\right)\frac{\Delta^4}{4!}\right.$$
$$\left.\left.+\cdots\right](B+dB)\right\}^+ \le \frac{1}{\Delta}\left\{\left[I\Delta+A^+\frac{\Delta^2}{2!}+(A^+)^2\frac{\Delta^3}{3!}+(A^+)^3\frac{\Delta^4}{4!}+\cdots\right]dB_{\max}+\left[dA_{\max}\cdot\frac{\Delta^2}{2!}+\left(A^+\cdot dA_{\max}+dA_{\max}\cdot A^+\right.\right.\right.$$
$$\left.+(dA_{\max})^2\right)\frac{\Delta^3}{3!}+\left(A^+\cdot dA_{\max}\cdot A^+ + dA_{\max}\cdot A^+\cdot dA_{\max}+dA_{\max}\cdot(A^+)^2+(A^+)^2\cdot dA_{\max}+(dA_{\max})^2\cdot A^+ + A^+\cdot(dA_{\max})^2\right.$$
$$\left.\left.\left.+(dA_{\max})^3\right)\frac{\Delta^4}{4!}+\cdots\right](B^+ + dB_{\max})\right\} = \frac{1}{\Delta}\left\{\left[I\Delta+(A^+ + dA_{\max})\frac{\Delta^2}{2!}+(A^+ + dA_{\max})^2\frac{\Delta^3}{3!}+(A^+ + dA_{\max})^3\frac{\Delta^4}{4!}+\cdots\right](B^+ + dB_{\max})\right.$$
$$\left.-\left[I\Delta+A^+\frac{\Delta^2}{2!}+(A^+)^2\frac{\Delta^3}{3!}+(A^+)^3\frac{\Delta^4}{4!}+\cdots\right]B^+\right\} = \frac{1}{\Delta}\left[\int_0^\Delta e^{(A^+ + dA_{\max})\tau}\,d\tau(B^+ + dB_{\max}) - \int_0^\Delta e^{A^+\tau}\,d\tau B^+\right] \qquad \square$$

The proof of Theorem 4 now continues by observing that the uncertain closed-loop system may be written as

$$\rho x(t) = A_{\rho c}x(t) + dA_{\rho c}x(t) \tag{D3}$$

where

$$A_{\rho c} = \begin{cases} A + BFC & \text{continuous} \\ A_\delta + B_\delta F_\delta C & \text{discrete} \end{cases}$$

and

$$dA_{\rho c} = \begin{cases} dA + dBFC & \text{continuous} \\ dA_\delta + dB_\delta F_\delta C & \text{discrete} \end{cases}$$

which has a solution given by Middleton and Goodwin[6]

$$x(t) = E(A_{\rho c},t)x(0) + \mathop{S}_{0}^{t} E(A_{\rho c},t-\tau-\Delta)\,dA_{\rho c}x(\tau)\,d\tau \tag{D4}$$

where

$$E(A_{\rho c},t) = \begin{cases} e^{A_{\rho c}t} & \text{continuous time} \\ (I+A_{\rho c}\Delta)^{t/\Delta} & \text{discrete time} \end{cases} \tag{D5}$$

$$\mathop{S}_{t_1}^{t_2} f(\tau)\,d\tau = \int_{\tau=t_1}^{\tau=t_2} f(\tau)\,d\tau \qquad \text{continuous time} \tag{D6a}$$

$$\mathop{S}_{t_1}^{t_2} f(\tau)\,d\tau = \Delta\sum_{k=t_1/\Delta}^{k=(t_2/\Delta)-1} f(k\Delta) \qquad \text{discrete time} \tag{D6b}$$

Next, use the real, positive, diagonal transformation

$$x(t) = D^{-1}z(t) \tag{D7}$$

and the properties shown in Ref. 6 that

$$E(DA_{\rho c}D^{-1}t) = DE(A_{\rho c},t)D^{-1} \tag{D8}$$

and

$$E(A_{\rho c},t) = ME(\Lambda_\rho,t)M^{-1} = \sum_{i=1}^{n} v_iw_i^*E(\xi_i,t) \tag{D9}$$

to obtain

$$z(t) = D\sum_{i=1}^{n} v_iw_i^*E(\xi_i,t)D^{-1}z(0) + \mathop{S}_{0}^{t} D\sum_{i=1}^{n} v_iw_i^*E(\xi_i,t-\tau-\Delta)\,dA_{\rho c}D^{-1}z(\tau)\,d\tau \tag{D10}$$

where $M$ is a modal matrix of $A_{\rho c}$; $\xi_i$ is the $i$th eigenvalue of $A_{\rho c}$ given by

$$\xi_i = \begin{cases} \lambda_i & \text{continuous time} \\ \dfrac{1}{\Delta}(e^{\lambda_i\Delta}-1) & \text{discrete time} \end{cases} \tag{D11}$$

with $v_i$ and $w_i^*$ the corresponding right and left eigenvectors, respectively; $\Lambda_\rho$ is a diagonal matrix with the $\xi_i$ on the diagonal; and $(\cdot)^*$ denotes complex conjugate transpose.

Note that $\|z(t)\| \to 0$ implies that $\|x(t)\| \to 0$. Next, we apply the absolute value operator, denoted by $(\cdot)^+$, to both sides of Eq. (D10) where "+" and "≤" are applied element by element to vectors and matrices.

$$z^+(t) \le \left[D\sum_{i=1}^{n} v_iw_i^*E(\xi_i,t)D^{-1}z(0)\right]^+ + \left[\mathop{S}_{0}^{t} D\sum_{i=1}^{n} v_iw_i^*E(\xi_i,t-\tau-\Delta)\,dA_{\rho c}D^{-1}z(\tau)\,d\tau\right]^+ \tag{D12}$$

$$z^+(t) \le D\sum_{i=1}^{n}(v_iw_i^*)^+E^+(\xi_i,t)D^{-1}z^+(0) + \mathop{S}_{0}^{t} D\sum_{i=1}^{n}(v_iw_i^*)^+E^+ \times(\xi_i,t-\tau-\Delta)\,dA_{\rho c\max}D^{-1}z^+(\tau)\,d\tau \tag{D13}$$

where

$$dA_{\rho c\max} = \begin{cases} dA_{\max} + dB_{\max}(FC)^+ & \text{continuous time} \\ dA_{\delta\max} + dB_{\delta\max}(F_\delta C)^+ & \text{discrete time} \end{cases} \tag{D14}$$

with $dA_{\delta\max}$ and $dB_{\delta\max}$ given by Lemma D1; and where

$$E(\xi_i,t) = \begin{cases} e^{\lambda_i t} & \text{continuous time} \\ (1+\Delta\gamma_i)^{t/\Delta} & \text{discrete time} \end{cases} \tag{D15}$$

$$E^+(\xi_i,t) = \begin{cases} e^{\mathrm{Re}(\lambda_i)\cdot t} = e^{-\alpha_i t} & \text{continuous time} \\ \left[(1+\Delta\gamma_i)^+\right]^{t/\Delta} = e^{-\alpha_i k\Delta} & \text{discrete time} \end{cases} \tag{D16}$$

and $\alpha_i = -\mathrm{Re}(\lambda_i)$.

Next, apply the $S$ operator to both sides of Eq. (D13) to obtain

$$\underset{0}{\overset{t}{S}} z^+(t)\,dt \le D\sum_{i=1}^{n}(v_iw_i^*)^+\underset{0}{\overset{\infty}{S}}E^+(\xi_i,t)\,dt D^{-1}z^+(0) + \underset{t=0}{\overset{\infty}{S}}\underset{\tau=0}{\overset{t}{S}} D\sum_{i=1}^{n}(v_iw_i^*)^+E^+(\xi_i,t-\tau-\Delta) \times dA_{\rho c\max}D^{-1}z^+(\tau)\,d\tau\,dt \tag{D17}$$

where

$$\underset{0}{\overset{t}{S}}E^+(\xi_i,t)\,dt = \begin{cases} \displaystyle\int_0^\infty e^{\mathrm{Re}(\lambda_i)t}\,dt = \int_0^\infty e^{-\alpha_i t}\,dt = \frac{1}{\alpha_i} & \text{continuous time} \\ \displaystyle\Delta\sum_{k=0}^{\infty}\left[(1+\Delta\gamma_i)^+\right]^k = \Delta\sum_{k=0}^{\infty}e^{-\alpha_i k\Delta} = \frac{\Delta}{1-\exp(-\alpha_i\Delta)} & \text{discrete time} \end{cases} \tag{D18}$$

Consider the double $S$ term in Eq. (D17) which may be written as

$$\lim_{R\to\infty}\left[\underset{t=0}{\overset{R}{S}}\underset{\tau=0}{\overset{t}{S}} D\sum_{i=1}^{n}(v_iw_i^*)^+E^+(\xi_i,t-\tau-\Delta)\,dA_{\rho c\max}D^{-1}z^+ \times(\tau)\,d\tau\,dt\right] \tag{D19}$$

In continuous time, Eq. (D19) becomes

$$\lim_{R\to\infty}\left[\int_{t=0}^{R}\int_{\tau=0}^{t} D\sum_{i=1}^{n}(v_iw_i^*)^+e^{-\alpha_i(t-\tau)}\left[dA_{\max}+dB_{\max}(FC)^+\right] \times D^{-1}z^+(\tau)\,d\tau\,dt\right] \tag{D20a}$$

The order of integration in Eqs. (D20) may be interchanged because all of the functions inside the integrals are continuous functions of $t$ and $\tau$. Thus, Eq. (D20a) is equal to

$$\lim_{R\to\infty}\left[\int_{\tau=0}^{R}\int_{t=\tau}^{R} D\sum_{i=1}^{n}(v_iw_i^*)^+e^{-\alpha_i(t-\tau)}\,dA_{c\max}D^{-1}z^+ \times(\tau)\,dt\,d\tau\right]; \qquad 0\le\tau\le t \tag{D20b}$$

where $dA_{c\max}=dA_{\max}+dB_{\max}(FC)^+$.

Now use the change of variables given by $\gamma=t-\tau$, $d\gamma=dt$, and $\gamma\ge 0$. Then, Eq. (D20b) is equal to

$$\lim_{R\to\infty}\left[\int_{\tau=0}^{R}\int_{\gamma=0}^{R-\tau} D\sum_{i=1}^{n}(v_iw_i^*)^+e^{-\alpha_i\gamma}\,dA_{c\max}D^{-1}z^+(\tau)\,d\gamma\,d\tau\right] \le \lim_{R\to\infty}\left[\int_{\tau=0}^{R}\int_{\gamma=0}^{R} D\sum_{i=1}^{n}(v_iw_i^*)^+e^{-\alpha_i\gamma}\,dA_{c\max}D^{-1}z^+(\tau)\,d\gamma\,d\tau\right]$$

$$= \lim_{R\to\infty}\left[\int_{\gamma=0}^{R} D\sum_{i=1}^{n}(v_iw_i^*)^+e^{-\alpha_i\gamma}\,dA_{c\max}D^{-1}\,d\gamma\int_{\tau=0}^{R}z^+(\tau)\,d\tau\right] \le \lim_{R\to\infty}\left[D\sum_{i=1}^{n}\frac{(v_iw_i^*)^+}{-\alpha_i}(e^{-\alpha_iR}-1)\,dA_{c\max}D^{-1}\int_0^\infty z^+(\tau)\,d\tau\right]$$

$$0\le\tau\le t,\qquad \gamma\ge 0,\qquad \alpha_i>0 \tag{D20c}$$

Evaluating the limit of the term outside the integral in Eq. (D20c), note that $\tau$ is now a dummy variable of integration, recall that the $\alpha_i$ are positive to obtain

$$\le D\sum_{i=1}^{n}\frac{(v_iw_i^*)^+}{\alpha_i}\left[dA_{\max}+dB_{\max}(FC)^+\right]D^{-1}\int_0^\infty z^+(t)\,dt \tag{D21}$$

In discrete time, let $t=k\Delta$ and $\tau=j\Delta$. Then, Eq. (D19) becomes

$$\lim_{R\to\infty}\left[\Delta\sum_{k=0}^{(R/\Delta)-1}\Delta\sum_{j=0}^{k-1}D\sum_{i=1}^{n}(v_iw_i^*)^+\left[(1+\Delta\gamma_i)^+\right]^{k-j-1} \times dA_{\delta c\max}D^{-1}z^+(j)\right] \tag{D22}$$

where $dA_{\delta c\max}=dA_{\delta\max}+dB_{\delta\max}(F_\delta C)^+$.

On changing the order of the summations, Eq. (D22) becomes

$$\lim_{R\to\infty}\left[\Delta^2\sum_{j=0}^{(R/\Delta)-2}\sum_{k=j+1}^{(R/\Delta)-1}D\sum_{i=1}^{n}(v_iw_i^*)^+\left[(1+\Delta\gamma_i)^+\right]^{k-j-1} \times dA_{\delta c\max}D^{-1}z^+(j)\right] \tag{D23a}$$

Then, let $q=k-j-1$ to obtain

$$\lim_{R\to\infty}\left[\Delta^2\sum_{j=0}^{(R/\Delta)-2}\sum_{q=0}^{(R/\Delta)-j-2}D\sum_{i=1}^{n}(v_iw_i^*)^+ \times\left[(1+\Delta\gamma_i)^+\right]^q dA_{\delta c\max}D^{-1}z^+(j)\right] \tag{D23b}$$

$$\le \lim_{R\to\infty}\left[\Delta^2\sum_{j=0}^{R/\Delta}\sum_{q=0}^{R/\Delta}D\sum_{i=1}^{n}(v_iw_i^*)^+ \times\left[(1+\Delta\gamma_i)^+\right]^q dA_{\delta c\max}D^{-1}z^+(j)\right] \tag{D24}$$

$$\le \lim_{R\to\infty}\left[\Delta^2D\sum_{i=1}^{n}(v_iw_i^*)^+\sum_{q=0}^{R/\Delta}\left[(1+\Delta\gamma_i)^+\right]^q \times dA_{\delta c\max}D^{-1}\sum_{j=0}^{R/\Delta}z^+(j)\right] \tag{D25}$$

$$\le \lim_{R\to\infty}\left[\Delta^2D\sum_{i=1}^{n}(v_iw_i^*)^+\left(\frac{1-\left[(1+\Delta\gamma_i)^+\right]^{(R/\Delta)+1}}{1-(1+\Delta\gamma_i)^+}\right) \times dA_{\delta c\max}D^{-1}\sum_{j=0}^{R/\Delta}z^+(j)\right] \tag{D26}$$

Use Eq. (D11) to obtain $(1+\Delta\gamma_i)=\exp(\lambda_i\Delta)$. Hence, $(1+\Delta\gamma_i)^+=\exp(-\alpha_i\Delta)$ and

$$\lim_{R\to\infty}\left[1-\left[(1+\Delta\gamma_i)^+\right]^{(R/\Delta)+1}\right] = \lim_{R\to\infty}\left[1-\exp\left[-\alpha_i\Delta(R/\Delta+1)\right]\right]=1$$

because $\alpha_i>0$. Substitute this result into Eq. (D26) to obtain

$$\text{Eq. (D26)}\le\Delta^2D\sum_{i=1}^{n}\frac{(v_iw_i^*)^+}{1-(1+\Delta\gamma_i)^+}\,dA_{\delta c\max}D^{-1}\sum_{j=0}^{\infty}z^+(j) \tag{D27}$$

## Appendix E: State Space Matrices for the EMRAAT Missile

$$A = \begin{bmatrix} -0.5007 & -0.9845 & 0.1736 & 0 & 0.109 & 0.0691 & 0 \\ 16.83 & -0.5748 & 0.01233 & 0 & -132.8 & 27.19 & 0 \\ -3227.0 & 0.3208 & -2.0990 & 0 & -1620.0 & -1240.0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -179 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -179 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

Combine Eqs. (D21) and (D27) and use Eq. (D6) to obtain

$$\int_{t=0}^{\infty}\int_{\tau=0}^{t} D\sum_{i=1}^{n}(v_iw_i^*)^+E^+(\xi_i,t-\tau-\Delta)\,\mathrm{d}A_{\rho c\max}D^{-1}z^+(\tau)\,\mathrm{d}\tau\,\mathrm{d}t$$

$$\leq D\sum_{i=1}^{n}\frac{(v_iw_i^*)^+}{f(\xi_i)}\,\mathrm{d}A_{\rho c\max}D^{-1}\int_0^{\infty}z^+(t)\,\mathrm{d}t \tag{D28}$$

where

$$f(\xi_i)=\begin{cases} -\mathrm{Re}(\lambda_i)=\alpha_i & \text{continuous time} \\ \dfrac{1}{\Delta}\left[1-(1+\Delta\gamma_i)^+\right]=\dfrac{1}{\Delta}\left[1-e^{-\alpha_i\Delta}\right] & \text{discrete time} \end{cases} \tag{D29}$$

Substitute Eq. (D28) into Eq. (D17) and use Eq. (D18) to obtain

$$\int_0^{\infty}z^+(t)\,\mathrm{d}t\leq D\sum_{i=1}^{n}\frac{(v_iw_i^*)^+}{f(\xi_i)}D^{-1}z^+(0)$$

$$+D\sum_{i=1}^{n}\frac{(v_iw_i^*)^+}{f(\xi_i)}\,\mathrm{d}A_{\rho c\max}D^{-1}\int_0^{\infty}z^+(t)\,\mathrm{d}t \tag{D30}$$

Take norms in Eq. (D30) and rearrange to obtain

$$\left\|\int_0^{\infty}z^+(t)\,\mathrm{d}t\right\|\leq\frac{\left\|D\sum_{i=1}^{n}\frac{(v_iw_i^*)^+}{f(\xi_i)}D^{-1}z^+(0)\right\|}{1-\left\|D\sum_{i=1}^{n}\frac{(v_iw_i^*)^+}{f(\xi_i)}\,\mathrm{d}A_{\rho c\max}D^{-1}\right\|} \tag{D31}$$

Thus,

$$\left\|\int_0^{\infty}z^+(t)\,\mathrm{d}t\right\|<\infty$$

which implies

$$\int_0^{\infty}\|z^+(t)\|\,\mathrm{d}t<\infty$$

if

$$\left\|D\sum_{i=1}^{n}\frac{(v_iw_i^*)^+}{f(\xi_i)}\,\mathrm{d}A_{\rho c\max}D^{-1}\right\|<1 \tag{D32}$$

Note that in continuous time, $x(t)$ and $z(t)$ are uniformly continuous on $(0,\infty)$ because of the linearity of the uncertain closed-loop plant that together with Eq. (D32) and a theorem in Ref. 12 implies that $\|z^+(t)\|\to 0$ as $t\to\infty$. This implies that $\|x(t)\|\to 0$ as $t\to\infty$ which proves that the linear uncertain closed-loop plant is asymptotically stable. In discrete time, the same result follows without the need for uniform continuity.

Finally, Perron weightings may be used for the matrix $D$ in Eq. (D32) in the same manner as shown by Sobel et al.[4] for an earlier robustness result. Thus, to reduce conservatism, Eq. (D32) may be replaced by

$$\lambda_{\max}\left\{\sum_{i=1}^{n}\frac{(v_iw_i^*)^+}{f(\xi_i)}\,\mathrm{d}A_{\rho c\max}\right\}<1 \tag{D33}$$

where $\lambda_{\max}(\cdot)$ of a nonnegative matrix denotes the real nonnegative eigenvalue $\lambda_{\max}\geq 0$ such that $\lambda_{\max}\geq|\lambda_i|$ for all eigenvalues $\lambda_i$.

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & 179 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 179 & 0 \end{bmatrix}^T$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$


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## References


[1] Sobel, K. M., and Cloutier, J. R., "Eigenstructure Assignment for the Extended Medium Range Air-to-Air Missile," *Journal of Guidance, Control, and Dynamics,* Vol. 15, No. 2, 1992, pp. 529–531.

[2] Andry, A. N., Jr., Shapiro, E. Y., and Chung, J. C., "Eigenstructure Assignment for Linear Systems," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. AES-19, Sept. 1983, pp. 711–729.

[3] Bossi, J. A., and Langehough, M. A., "Multivariable Autopilot Designs for a Bank-To-Turn Missile," *Proceedings of the American Control Conference,* American Control Council, Green Valley, AZ, Vol. 1, 1988, pp. 567–572.

[4] Sobel, K. M., Banda, S. S., and Yeh, H.-H., "Robust Control for Linear Systems with Structured State Space Uncertainty," *International Journal of Control,* Vol. 50, May 1989, pp. 1991–2004.

[5] Yu, W., Piou, J. E., and Sobel, K. M., "Robust Eigenstructure Assignment for the Extended Medium Range Air to Air Missile," *Proceedings of the 30th IEEE Conference on Decision and Control,* Inst. of Electrical and Electronics Engineers, Piscataway, NJ, Vol. 3, 1991, pp. 2976–2981.

[6] Middleton, R. H., and Goodwin, G. C., *Digital Control and Estimation: A Unified Approach,* Prentice-Hall, Englewood Cliffs, NJ, 1990, pp. 65, 66, 81–84.

[7] Grace, A., *Optimization Toolbox for Use with MATLAB™*, Mathworks, Natick, MA, 1990.

[8] Middleton, R. H., *Delta Toolbox for Use with MATLAB™*, Mathworks, Natick, MA, 1990.

[9] Langehough, M. A., and Simons, F. E., "6DOF Simulation Analysis for a Digital Bank to Turn Autopilot," *Proceedings of the American Control Conference,* American Control Council, Green Valley, AZ, Vol. 1, 1988, pp. 573–578.

[10] "Military Specification-Flying Qualities of Piloted Airplanes," MIL-F-8785C, ASD/ENESS, Wright-Patterson AFB, OH, Nov. 1980.

[11] McRuer, D., Ashkenas, I., and Graham, D., *Aircraft Dynamics and Automatic Control,* Princeton Univ. Press, Princeton, NJ, 1973, pp. 255–259.

[12] Hsu, J. C., and Meyer, A. U., *Modern Control Principles and Applications,* McGraw-Hill, New York, 1968, pp. 727, 728.